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LETTER TO THE EDITOR

Maximum weight vectors possess minimal uncertainty

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Abstract. An appropriate uncertainty measure for a compact Lie group is the invariant dispersion

$$(\Delta F)^2 = \langle g^{rs}(F_r - \langle F_r\rangle)(F_s - \langle F_s\rangle)\rangle.$$

We prove that it is minimised for maximum weight vectors (of greatest length in the weight space), and those unitarily related to them.

In a recent paper (Delbourgo 1977) we examined the problem of how to define quasiclassical states (characterised by a least uncertainty) for O(3) and closely associated groups. Our investigation showed that the most natural measure of indeterminancy, the invariant dispersion,

$$(\Delta J)^2 \equiv \langle \boldsymbol{J}^2\rangle - \langle \boldsymbol{J}\rangle \,.\, \langle \boldsymbol{J}\rangle,$$

was minimised for maximum weight angular momentum states

$$\boldsymbol{J}\,.\,\boldsymbol{n}|jj\rangle_{\boldsymbol{n}} = j|jj\rangle_{\boldsymbol{n}},$$

and we guessed that the result carried over to arbitrary compact Lie groups; namely, the maximum weight vectors corresponded most closely to the classical situation of absolute precision. In this letter we would like to outline a simple proof of this conjecture.

Let F_s denote the set of generators of a compact Lie algebra obeying the commutation rules

$$[F_r, F_s] = \mathrm{i}\, C_{rs}{}^t F_t.$$

The positive-definite Cartan metric

$$g_{rs} = \tfrac{1}{2} C_{rp}{}^q C_{qs}{}^p,$$

and its inverse g^{rs}, can be used to construct the quadratic Casimir $F^2 \equiv g^{rs}F_rF_s$. Following our O(3) analysis we contend that the most appropriate measure of quantum indeterminancy is the invariant dispersion (variance)

$$(\Delta F)^2 \equiv \langle(\Delta\hat{F})^2\rangle \equiv \langle g^{rs}(F_r - \langle F_r\rangle)(F_s - \langle F_s\rangle)\rangle = \langle g^{rs}(F_rF_s - \langle F_r\rangle\langle F_s\rangle)\rangle. \qquad (1)$$

We will now show that eigenvectors of the Casimirs $F^2, F^3, \ldots$ have least ΔF when they are of maximum weight (i.e. have weight vectors of maximum length).

First we choose a canonical basis whereupon $g^{rs} \propto \delta^{rs}$, and make the conventional split into Cartan operators H_k (defining the rank) and $E_{\pm\alpha}$ (changing the weights). As $(\Delta\hat{F})^2$ is a positive-definite operator, it has a lowest eigenvalue, and in that lowest eigenstate the lowest expectation value is attained. The minimum dispersion states are therefore among the eigenstates of $(\Delta \hat{F})^2$, satisfying

$$(F_s F_s - 2F_s\langle F_s\rangle_\psi + \langle F_s\rangle_\psi \langle F_s\rangle_\psi)|\psi\rangle \propto |\psi\rangle.$$

But $|\psi\rangle$ is taken to be already an eigenvector of $F_s F_s$ (among other possible Casimirs). Therefore a necessary condition for minimal dispersion is

$$F_s\langle F_s\rangle_\psi|\psi\rangle \propto |\psi\rangle. \tag{2}$$

To find solutions of (2), first suppose that $|\psi\rangle$ is an eigenvector $|\boldsymbol{h}\rangle$ of the (rank) r commuting generators H_k which form a basis for the Cartan sub-algebra. Then

$$\begin{aligned} &H_k|\boldsymbol{h}\rangle = h_k|\boldsymbol{h}\rangle, && k=1,\ldots,r,\\ &\langle \boldsymbol{h}|E_\alpha|\boldsymbol{h}\rangle = 0, && \text{all roots } \alpha, \end{aligned} \tag{3}$$

because of the step-up and -down action of the E_α. For generators F_s chosen so that r of them are the H_k (the others being Hermitian linear combinations of the E_α),

$$F_s\langle F_s\rangle_h|\boldsymbol{h}\rangle = H_k\langle H_k\rangle_h|\boldsymbol{h}\rangle = h_k h_k|\boldsymbol{h}\rangle$$

showing that any state $|\boldsymbol{h}\rangle$ solves (2), and ΔF is minimised when $\langle F_s\rangle\langle F_s\rangle$ is maximised, i.e. for greatest $|\boldsymbol{h}|^2 \equiv h_k h_k$; namely for *vectors of maximum weight.*

The second part of the proof consists in finding the most general solutions of (2). For this we note that it is always possible to find a group transformation such that $F_s n_s$, with $\boldsymbol{n}$ a unit vector, is transformed into an element $H_k a_k$ in the Cartan sub-algebra, where $\boldsymbol{a}$ is a unit vector in r-dimensional space, i.e.

$$U(\boldsymbol{n})F_s n_s U^{-1}(\boldsymbol{n}) = H_k a_k(\boldsymbol{n})$$

where $U(\boldsymbol{n})$ is an operator† lying in the coset space G/H. The vectors $|\boldsymbol{h}\rangle$ are eigenvectors of this expression, so

$$F_s n_s U^{-1}(\boldsymbol{n})|\boldsymbol{h}\rangle = h_k a_k(\boldsymbol{n})U^{-1}(\boldsymbol{n})|\boldsymbol{h}\rangle$$

or

$$\frac{F_s\langle F_s\rangle_\psi}{(\langle F_t\rangle_\psi\langle F_t\rangle_\psi)^{1/2}}|\psi\rangle = h_k a_k(\boldsymbol{n})|\psi\rangle \tag{4}$$

if we set $U^{-1}(\boldsymbol{n})|\boldsymbol{h}\rangle = |\psi\rangle$ and choose $n_s = \langle F_s\rangle_\psi/(\langle F_t\rangle_\psi\langle F_t\rangle_\psi)^{1/2}$. For this choice of $\boldsymbol{n}$, $a_k = h_k/|\boldsymbol{h}|$, as can be seen by taking the inner product of (4) with $|\psi\rangle$, yielding

$$(\langle F_s\rangle_\psi\langle F_s\rangle_\psi)^{1/2} = h_k a_k.$$

But the left-hand side as an invariant under group transformations and is equal to $h_k h_k/|\boldsymbol{h}|$ when $U(\boldsymbol{n})=1$, i.e. $|\psi\rangle = |\boldsymbol{h}\rangle$, so that $a_k = h_k/|\boldsymbol{h}|$. Therefore $U^{-1}|\boldsymbol{h}\rangle$ solves (2) with eigenvalue $h_k h_k$. These vectors are none other than Perelomov's coherent states (Perelomov 1972).

Finally, to minimise ΔF, we need to maximise $\langle F_s\rangle\langle F_s\rangle$, and this is achieved by taking $|\boldsymbol{h}\rangle$ to be a maximum weight state. The foregoing shows that any vector $U|\boldsymbol{h}\rangle$ unitarily related to a maximum weight state equally well minimises ΔF. This is, of course, to be expected, as the choice of the sub-algebra basis $(H_1,\ldots,H_r)$ is not

† For instance, if we are dealing with O(N) and the rotation operators $J_{\mu\nu}$, $UJ_{\mu\nu}n^{\mu\nu}U^{-1} = J_{12}a^{12} + J_{34}a^{34} + \ldots$ and U is an orthogonal transformation which rotates the special tensor a into n. Similarly, if G is the group SU(N), $UF_a^b n_b^a U^{-1} = H_1^1 a_1^1 + H_2^2 a_2^2 + \ldots$ where U is a unitary transformation.

unique and the maximum weight eigenstates of any unitarily equivalent basis (the states $U|\boldsymbol{h}\rangle$) should also minimise ΔF.

For illustration take SU(3) and representations $\Phi^{\{b_1 \ldots b_q\}}_{\{a_1 \ldots a_p\}}$ labelled by the pair (p, q). In this case the least

$$\Delta F \equiv (\tfrac{1}{2}\langle F^a_b F^b_a \rangle - \tfrac{1}{2}\langle F^a_b \rangle \langle F^b_a \rangle)^{1/2}$$

equals $(p+q)^{1/2}$ and occurs, of course, for highest weight vectors like $\Phi^{\{2 \ldots 2\}}_{\{1 \ldots 1\}}$. The value of $\langle F^b_a \rangle \langle F^c_b \rangle \langle F^a_c \rangle$ is thereby completely determined and there is no point in considering uncertainties related to the cubic Casimir.

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